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by

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Algebra Readiness:
Building Strong Foundations Through Conceptual Understanding

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Building Strong Foundations Through Conceptual Understanding

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Report

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Dedication

It has been a long hard road to reach this point in my life. Words cannot adequately express my gratitude for my family and friends who have supported me through this process. I have learned much about persistence, resilience and overcoming challenges; I learned more about myself than about teaching mathematics. I thank my children, Austin, Skyler, Jordan, Mackenzie and Madison as they have been the guiding force that has helped me persevere. I thank all of the individuals in my cohort, especially Katherine Garrett, Jodi Wheeler, Stephanie Foster, and Darlene Sugarek, who made staying up late to work on math problems fun again. I thank Dr. Margarita Arellano at the Dean of Students Office at Texas State University-San Marcos, for her ethical leadership, guidance and support through these past few years. I thank Dr. Daniels for keeping me focused on what was important. I am grateful for his advice, and for challenging me to continually improve. I thank Dr. Armendariz for sharing his positive energy and love of mathematics with all of us. Finally, I thank Barbara Mockler, Leigh Germann, Carol Call, and Lisa Felizi for being my support system through some very difficult times in my life.

Mac Anderson once said, that "Life isn't about waiting for the storm to pass, but about learning to dance in the rain." This is has become my motto. Thanks to all those who have helped me dance in the rain.

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Abstract

Algebra Readiness:
Building Strong Foundations Through Conceptual Understanding

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Algebra readiness has been a topic of much research and debate. This report summarizes findings pertaining to how researchers and experts in the field of mathematics education define the term algebra readiness. The Southern Regional Education Board (SREB) identified a list of readiness indicators for success in algebra. This list is explored. Finally, current teaching practices that have been found successful in helping students become algebra ready are detailed.

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INTRODUCTION

The idea of "algebra for all" has been the resounding quest in the United States in recent years. With hopes of increasing the quantitative literacy of its citizens, the United States government has launched a national campaign to raise academic standards, particularly in the areas of mathematics. College readiness initiatives have been launched in nationwide hopes to prepare students to enter college with the academic skills to succeed. Since algebra is considered the "…gatekeeper to more advanced mathematics and opportunities" states have been raising standards and focusing on preparing students for a challenging educational path that includes algebra for all [**6**, p. 1]. Researchers and educators alike are seeking ways to answer the call of teaching algebra effectively, many coming up with solutions and alternatives in practices to help more students succeed.

Millions of dollars have been donated to schools to "fix" the problems associated with mathematics instruction and strengthen academic programs. Within state mandated performance assessments, many teachers and school administrators are facing significant challenges to meet the increasing demands of raising the standards in mathematics. Most recognize the limited time and resources available to teach students from multiple socio-economic, cultural and educational backgrounds.   As schools struggle to meet the demands, educators wonder if there is time to prepare students for algebra while still teach the grade level standards they are required to teach.

The case for algebra readiness is strong, as the demand for stronger skills in mathematics sounds loud and clear from colleges and the businesses across the world.

Still, debates abound as to how early children can be exposed to algebraic thinking, what fundamentals are required to be mastered before it is even feasible to introduce algebra to students. Finally, what are the best strategies to use to help students become algebraic thinkers in today's world? As researchers continue to conduct studies to answer these questions, the call for algebra readiness is hard to miss.

This paper shares findings on algebraic thinking, readiness indicators for learning and understanding algebra, Piagetian and neoPiagetian theories of cognitive development. In addition, research on both identifying and crossing the cognitive gap between arithmetic and algebra is included in efforts to establish the premise for teaching algebra with integrity and reality. Finally, implications and pedagogical recommendations are integrated.

Indeed, algebra is the gatekeeper for many future successes. This being the case, it is critical to establish strong a strong mathematics foundation in our students to help them succeed. No longer is it sufficient to know solely the basics. It is now essential to include pattern recognition, generalizations, abstract thinking and logic into our tool belts. Only then can the students of today be properly prepared for the demands of tomorrow.

Chapter 1: Just What Is Algebraic Thinking?

The National Council of Teachers of Mathematics urges "algebraic thinking in all elementary grades" [**14**, p. 163]. In research related to student work with patterns, differences have been identified related to inductive reasoning, pattern recognition, and arithmetic, bringing about the question, just what is *algebraic thinking*? For some,

> …algebraic thinking has become a catch-all phrase for the mathematics teaching and learning that will prepare students with the critical thinking skills needed to fully participate in … successful experiences in algebra. [**6,** p. 1]

Most experts in the field agree that algebraic thinking consists of problem solving techniques, pattern recognition and representation, and quantitative reasoning [**6**]. The ability to make generalizations and express mathematical thought processes through symbolic representations is at "the very heart of algebra" [**14**, p. 163]. Algebraic thinking is a "way of thinking" that involves proper use of signs and symbols to identify patterns, mathematical models, structure and functional relationships [**3**]. Algebraic thinking differs from arithmetical thinking primarily due to the fact that in algebra, symbolic representations are more commonly used than numerical approaches to solving problems.

Table 1: Components of Algebraic Thinking. [**6,** p. 2]

Mathematical Thinking Tools Components of Algebraic Thinking	
Mathematical Thinking Tools	**Fundamental Algebraic Ideas**
Problem solving skills • Using problem solving strategies • Exploring multiple approaches/solution paths Representation skills • Displaying relationships visually, symbolically, numerically, verbally • Translating among different representations • Interpreting information within representations Quantitative reasoning skills • Analyzing problems to extract and quantify • Inductive and deductive reasoning	Algebra as generalized arithmetic • Conceptually based computational strategies • Ratio and proportion • Estimation Algebra as the language of mathematics • Meaning of variables and variable expressions • Meaning of solutions • Understanding and using properties of the number system • Reading, writing, manipulating numbers and symbols using algebraic conventions • Using equivalent symbolic representations to manipulate formulas, expressions, equations, inequalities Algebra as a tool for functions and mathematical modeling • Seeking, expressing, generalizing patterns and rules in real world contexts • Representing mathematical ideas using equations, tables, graphs, or words • Working with input/output patterns • Developing coordinate graphing skills

Chapter 2: A Cognitive Gap Between Arithmetic and Algebra

In an effort to determine when students are algebra ready, Herscovics and Linchevski launched a research study to identify cognitive gaps and "upper limits" [**5,** p. 59**]** of students' conceptual understanding when solving basic algebraic equations without prior instruction. In this study, Herscovics and Linchevski argue that in order for teachers to adequately prepare students for success in algebra, teachers need to identify the scope and depth of students' understanding. For the purpose of this paper, the *cognitive gap* is the "demarcation between arithmetic and algebra" [**5**, p. 63]. Without first determining where the boundary line is between arithmetic and algebraic reasoning, Herscovics and Linchevski claim that students with earlier exposure to algebra will have limited understanding that may influence their algebra readiness.

The primary aim of this study was to solve the mystery of algebra preparedness and to answer the question: "Is algebra within the reach of most students?" [**5**, p. 60]. In an effort to find solutions, Herscovics and Linchevski sampled students from a nearby, high achieving school to determine to what extent some of the brightest students think "algebraically" without prior instruction on solving simple algebraic equations [**5**].

Herscovics and Linchevski formalized an agreement with the participating teachers to delay lessons involving working with variables until after their initial interview with the students. To collect data, Herscovics and Linchevski interviewed a class of seventh graders while the students attempted to solve 50 algebraic problems. During the interview, the students were presented with different types of algebraic equations and asked to use whatever method necessary to solve the equation. Though all of the equations were single variable, some included variables on both sides. Several problems included variables used twice on the same side of the equation [**5**].

Students were provided a calculator for basic computations, to remove any obstacles related to miscalculations and to ensure the study focused strictly on the algebraic reasoning. In addition, students were provided with scratch paper, and were encouraged to think aloud in order for the interviewers to accurately record students' attempts to solve the equation**.** Throughout the interview, the students were allowed to refer to the interviewer's notes to remind them of what they had said or had attempted to do to solve the problems. The assessment included a variety of equations where students were expected to apply the order of operations, inverse operations, and even strategies such as combining like terms to manipulate the variables and find the solution [**5**].

The results of this study showed students lacking in understanding of the meaning of the unknown. Students did not automatically assume that operations applied to numerals were equally allowed on the variables. In fact, most students either ignored the variable completely or simply combined and performed operations on the numerals, without affecting the variables. Examples of this being:

$$3n + 4 = 7n. \tag{1.1}$$

Further, after a series of investigations and interviews with students, Herscovics and Linchevski discovered that several students still had difficulty accepting "8 × a as the area of an indicated rectangle unless it was inserted in the formula 'Area of rectangle = 8 × a' [**5**, p. 63].

Consistent mistakes took place when trying to read and solve a problem such as:

$$364 = 796 - n, \tag{1.2}$$

where students read as "*n* minus 796 equals 364" and solved incorrectly. Problems in applying the order of operations correctly as well as issues with combining like terms were prevalent throughout the research study. Most students chose to ignore the variable and operated solely on the numbers when solving equations then simply attached the variable to their final response. In fact, "at no time did [Herscovics and Linchevski] see any evidence of students directly performing operations on or with the unknown" [**5**, p. 70].

During the interviews, the Herscovics and Linchevski discovered that there indeed exists a cognitive gap between arithmetic and algebra. Furthermore, it was clear that without prior instruction, many students did not know how to act upon the unknown using basic mathematics operations. There are many issues that make it difficult to adequately identify exactly where the demarcation line between arithmetic and algebra lies. The cognitive gap indicates an upper limit for students preparing for a formal algebra course. In fact, Herscovics and Linchevski claim that in addition to the cognitive gap they identified in their study, there are many others. Age appropriateness is a consideration as it accounts for a level of conceptualization required to think algebraically [**5**].

The impact of Herscovics' and Linchevski's research in today's middle and high school classrooms provides information pertaining to a "…transition phase between arithmetic and algebra" [**5**, p. 74]. According to the NCTM Standards, students "…from prekindergarten through grade 12 should be able to understand numbers, relationships among numbers, and number systems; understand meanings of operations and how they relate to one another" [**10**, para.1]. Since the structural properties of numbers are the same as that of the number system, students are often expected to easily transition from operating on numerals to operating on variables. By identifying the cognitive gap, pedagogical implications include a level of inquiry that involves teachers in careful design and development of arithmetic processes that can be more extensively used to prepare students for algebraic reasoning.

The National Council of Teachers of Mathematics has adopted standards that recommend algebra for all students [**10**]. Unfortunately, of the students who enroll in an algebra course, many "…fail or encounter major difficulties" [**5**, p. 60]. In such cases, Herscovics and Linchevski argue that the bridge between arithmetic and algebra is insufficient. Expecting students to have a natural ability or instinct to act upon the variable to solve problems without prior instruction is unreasonable [**5**].

Herscovics and Linchevski's study showed the need for teachers to recognize the cognitive gap and to provide a way for students to apply their knowledge of operations on numbers to unknowns. By conducting this study, and publishing the results, Herscovics and Linchevski hoped to provide teachers and teacher trainers with information pertaining to the cognitive gap between arithmetic and algebra. Results of the study provide teachers with information on how to bridge the gaps of understanding in students and provide areas of focus and considerations in lesson preparation and design. Taking into consideration the cognitive processes necessary for success in algebra, teachers can prepare their students and eliminate the gap that exists between arithmetic and algebra [**5**].

Chapter 3: Readiness Indicators for Algebraic Thinking

The Southern Regional Education Board (SREB) along with a panel of teachers and curriculum experts created a list of indicators for algebra readiness [**4**].

Table 2. Readiness Indicators [**4**, p. 13].

Readiness Indicators
1. Ability to solve real world problems 2. Use of mathematics language to explain thinking process 3. Use multiple problem solving approaches to identify correct answers and check for reasonableness of solutions; able to identify mistakes 4. Draws conclusions, creates arguments, and makes conjectures based on observations 5. Use of technology (including software and graphing calculators) to deepen level of understanding of mathematical concepts and ideas 6. Identify and describe differences between natural, whole, integers, real, rational and irrational numbers 7. Solve problems involving positive exponents, scientific notation, numerical factors, least common multiples, square roots and cube roots. 8. Use proportional reasoning to model and solve application problems involving rate, indirect variation, scale drawings and similar triangles. 9. Read, analyze and convert various measures given within application settings 10. Use multiple pieces of information to solve problems 11. Solve one-and two-step equations and inequalities in one variable 12. Determine the equation of a line and represent the equation in a variety of ways

Multiple studies continue to attempt to answer the question pertaining to when students are deemed “algebra ready” and even what the term implies. Educators whose training revolves around Piaget’s stages of development are reluctant to engage students in algebraic thinking too early [**4**]. Some U.S. educators do not think it is possible to teach students algebra who are not adept with the basics such as multiplication facts, fraction and integer operations while students of other countries often begin studying algebra before mastering arithmetic [**4**]. The challenge lies in convincing teachers to change their practice and discover ways to develop algebraic thinking throughout various stages of development.

According to Piaget, children at the concrete stage of development are capable of mental math provided it revolves around real objects. The transition from concrete operational to formal operational is expected to begin around age 11 and continue through age 15. Still, some studies have found that many adults never fully make the transition into the world of generalizations, abstract thinking and metacognition. If this is correct, then it is not surprising to see why so many children struggle with algebra [**4**]. Teachers who base their practice on Piaget’s stages of development, propose waiting to teach algebra until children reach the stage where they are developmentally ready for abstract thinking and are often resistant to an early introduction of algebra. In fact, even after 25 hours of professional development to 5th and 6th grade teachers in a California school district, a resounding 40% of the teachers surveyed claimed that children need to reach the abstract stage of development before mastery of algebra concepts can occur [**4].**

As more research is conducted on children in different tasks, and cross cultural settings, some findings claim children reach the concrete operations at the age of 8 or 9 while other cultures do not at move through the formal operations stages at all yet the children successfully learn algebra concepts at earlier stages of development than Piaget predicts [**4**]. As a result of these inconsistencies, developmental theorists consider the possibilities of learning along a continuum, whether individual differences and cognitive enrichment activities.

Early "…algebra success indicators such as experience, context, cultural traditions and language" [**4,** p. 5**]** have been identified as crucial to cognitive development. Connections between the new material and prior knowledge must be made to improve retention. According to the novice expert theory, exposing students to algebra concepts before the age of 13 or 14 can be beneficial in order for them to achieve mastery and could instead pave the road for algebra success in the future **[4**]. The arguments that mastery of arithmetic before algebra is required to be successful, is refuted by evidence of students in other countries understanding a substantial amount of algebra prior to mastering all of the arithmetic skills [**4].**

Key obstacles to algebra success exist. These include cognitive obstacles such as difficulty understanding the arithmetic sign system. One example of this is for an algebraic expression such as "4*x*" which represents the numeral "4" next to "*x*," where when x = 2, for example, is sometimes considered to be the numeral “42”. This very common mistake is evidence of lack of understanding among students as to the meaning of “4*x.*” Additional difficulties arise from lack of experience in generalizing numerical patterns. Examples include using different letters (*a, b, c)* instead of expressing the other unknowns in terms of the first such as (*m* - 1, *m, m* + 3) [**4].**

Changes in the ways solutions are found are among the shifts in problem solving strategies required to solve algebraic problems. For instance, for a problem such as:

> Daniel went to visit this grandmother, who gave him $1.50. Then he bought a book costing $3.20. If he has $2.30 left, how much money did he have before visiting his grandmother? [**4,** p.7]

To solve this problem, students in the U.S. use the equal sign to “…announce the next result: 2.30 + 3.20 = 5.50 – 1.50 = 4.00” [**4,** p. 7]. In algebra, the problem would involve incorporating a variable to find the missing value. Without any prior instruction in the use of this technique, students do not introduce the idea of a variable. According to Henry:

> The shift in thinking that students must make in order to move from solving problems in steps (in a means-end strategy of getting one step closer to the solution with each step), to setting up the entire problem first, is dramatic. [**4**, p. 7]

Still, this does not appear to be an issue related to maturation, but instead related to current teaching practices. In fact, cognitive discontinuity is also evident in students’ initial difficulty in manipulating algebraic expressions caused by a gap between procedural and structural understanding of algebra [**4**]. As practitioners rethink algebra prerequisites, it is agreed that many of the problems students have with algebra have little to do with inadequate arithmetic skills. Most problems relate to ways of thinking algebraically. For many teachers, algebra is about procedures and algorithms. When educational researchers look at algebra, students' difficulties are evident in their limited knowledge of "…sign systems, translations, and structural understanding" [**4**, p. 8]. Studies have proven that young students can learn to reason algebraically provided they are given a strong, conceptual foundation where their mathematical knowledge can flourish. Supporting the NCTM standards:

> All students should learn algebra. By viewing algebra as a strand in the curriculum from prekindergarten on, teachers can help students build a solid foundation of understanding and experience as a preparation for more sophisticated work in algebra…systematic experience with patterns can build up to an understanding of the idea of function, and experience with numbers and their properties lays a foundation for later work with symbols and algebraic expressions. [**11**, p. 37]

Chapter 4: Crossing the Cognitive Gap Between Arithmetic and Algebra

Upon discovering a cognitive gap between arithmetic and algebra, Linchevski and Herscovics conducted a follow up study to determine best ways to cross the cognitive gaps found in students, particularly regarding operating on the unknown. A series of teaching experiments followed to identify whether this gap could indeed be crossed. In arithmetic, it is possible to keep computations and objects separate; this is not the case in algebra. In fact, the dual nature of algebraic expressions, operational and structural, is problematic [**7**]. The additional dilemma of multiple interpretations of letters in algebra (the unknown, a "generalized number" [**7,** 42], as a variable of a function) pose a problem for students.

In designing teaching that uses equations like $n + n = 76$, the data showed that it was more relevant to students as it made it easier to see the dual nature of the unknown with a possible solution. Results showed that students were able to naturally group singletons in equations, establish equivalency, and use the equivalency in grouping like terms. For example, $6n + 18 = 8n + 4$ can be decomposed to $6n + 4 + 14 = 6n + 2n + 4$. With a simple cancellation technique, $14 = 2n$, thus $n = 7$ [**7**]. This cancellation procedure identified by Linchevski and Herscovics as "Cancellation within Equations" proved simpler for students than traditional mathematical procedures [**7**, p. 44]. To do this successfully, students needed to have the ability to compare expressions such as $25 + 365 – 51$ and $12 + 8 + 365 – 51$ [**7**]. Many students then resorted to "…systematic substitution…guess and check" [**7**, p. 47]. Once students were able to identify this strategy as "…solving using inverse operations in reverse order" [**7**, p. 47], they were able to use this procedure regularly to solve similar equations. Through questions such as

"In the equation $n + n + n + n = 112$, what would be a shorter way of writing the left hand side?" [**7**, p. 47], students were able to quickly identify $4n = 112$ [**7**]. Additional questions helped students deepen their level of understanding and expand their algebraic thinking. These were:

> "Can you group the sum of the left of $3n + 5n = 136$? …
>
> Do you think that we can add $3n$ and $5n$ even before we what the number is? ...
>
> Is $3n + 5n = 8n$ for every number n?" [**7**, p. 48]

Guided questions helped students work through the problem solving process, though some unexpected cognitive dilemmas did develop when solving problems such as:

(1) $102 = 22n - 17n + 49 – 12$ and

(2) $19n + 67 – 11n – 48 = 131$. [**7**, p. 51]

In (1), students felt the need to rewrite the problem with the expression on the left, and the solution on the right, in order to put the equation in the "correct order" [**7**, p. 51]. Equation (2) presented a different issue as students tried to solve this problem by grouping the $19n$ and the $11n$, neglecting the subtraction symbol in front of $11n$. Linchevski and Herscovics referred to this cognitive problem as "jumping off the posterior operation" [**7**, p. 51]. A different issue took place when two students opted to subtract "…48 from 67 but then rewrote the initial equation as $30n – 19 = 131$" [**7**, p. 51]. This problem was labeled "inability to select the appropriate operation for the partial sum" [**7**, p. 51].

An interesting discovery resulting from this study was that once students were taught that the equation could be reordered as in $19n – 11n + 67 - 48 =131$, where the like terms were consecutive, students were able to solve subsequent problems using correct grouping, applying inverse operations and finding the solutions [**7**]. Providing students detailed descriptions helped students cross some aspects of the cognitive gap by overcoming the initial "…hurdle created by the presence of a variety of arithmetic operations" [**7**, p. 52].

Since most physical models have some restrictions to their applicability that eventually lead to significant cognitive difficulties for students, Linchevski and Herscovics opted for an arithmetic model. Choosing the balance model as an introductory model for solving equations, student were then able to justify the solution process via "…decomposition and cancellation on the basis of an arithmetic model" [**7**, p. 53].

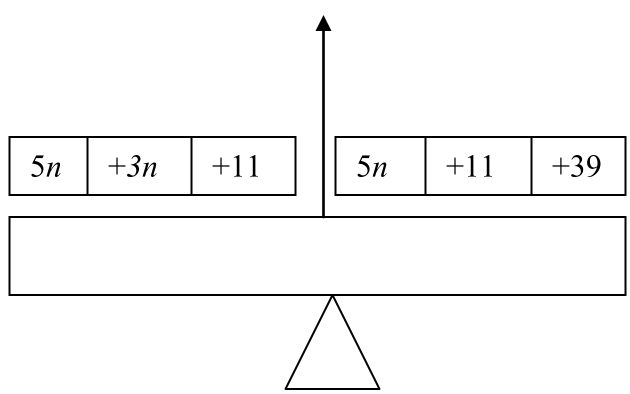

Figure 1: The balance model [**7**, p. 53].

Students were introduced to the balance model for solving the equation $5n + 3n + 11 = 5n + 11 + 39$ through cutouts of its different terms. These cutouts were put on the scale drawn on their activity sheet. Students were then asked to identify equal terms on both sides. As students worked through the problem, identification of equal terms led them to solve the problem correctly. Benefits of this process include an effective way to condense the cancellation process justifying it with an arithmetic model [**7**].

The cancellation procedure was introduced by directing students' attention to the algebraic equation constructed from the numerical one. Students were asked a series of guided questions such as: Can the equation $8n + 11 = 5n + 50$ be rewritten as $8n + 11 = 5n + 39 + 11$? If so, will the solution be the same? [**7**]. Subsequent questions scaffolded student learning and helped students develop a deeper understanding of the techniques needed to solve these equations.

<u>Summary of Solution Procedure</u>

$$
\begin{aligned}
8n + 11 &= 5n + 50 \\
8n + \cancel{11} &= 5n + \cancel{11} + 39 \\
8n &= 5n + 39 \\
\cancel{5n} + 3n + 11 &= \cancel{5n} + 39 \\
3n &= 39 \\
n &= 13
\end{aligned}
$$

Figure 2: Summary of solution procedure. [**7**, p. 55].

Students were encouraged to use the notation (Figure 2) to justify their problem solving approach. Calling this process "cancelling 11 on both sides" or "cancelling addition of 11 on both sides" [**7**, p. 55] assisted in establishing a common language for the process. As students practiced the procedure, a new cognitive problem arose. Difficulty in solving problems with singletons indicated that students did not understand that $1n = n$.   Once this obstacle was overcome, students were able to decompose $6n$ into $n + 5n$ [**7**].

Results of this study answered the "…questions about accessibility and limitations, and uncovered some significant cognitive obstacles …[such as] detachment of the minus sign [and] jumping off the posterior operation, … and students inability to select the appropriate operation for the partial sum in an equation" [**7**, p. 60; 61]. Pedagogical implications include how (1) teaching students the balance model and the decomposition model students would have "…some idea of what algebra is about; …(2) obstacles can be appropriate preparation in arithmetic; [and] (3) research involves the timing and cognitive value of pedagogical intervention" [**7**, p. 61]. Identifying students' thinking process through the individual interviews was one of the benefits of this study. Additionally, Linchevski and Herscovics built students conceptual understanding by helping students connect existing knowledge to new concepts [**7**].

All in all, though there were some limitations in assisting students in crossing the cognitive gap between arithmetic and algebra, this study proved that there are ways to guide students through it. Primarily, using an incremental approach is needed to help students successfully cross the gap and develop their algebraic thinking skills. If a generalized process is introduced too early, students either rejected or failed to use the procedure until they were able to make the necessary connections to solving algebraic problems [**7**].

Chapter 5: Algebra with Integrity and Reality

Algebra is about the "…basic number systems, …arithmetic operations on these numbers systems, …linear ordering and resulting geometric structure,… [along with] the study of algebraic equations that arise naturally in these systems" [**3**, p. 9]. Kriegler considers algebra in three ways: generalized arithmetic, as a language, as a tool for mathematical modeling and functions. According to Kriegler, it is difficult to apply mathematical thinking tools and logic with "nothing to think about (algebraic ideas)" [**6**, p. 1]. Using mathematical thinking tools such as problem solving, representation and reasoning are crucial in daily life. An individual's ability to solve problems, make connections and reason through complex situations can be a powerful tool for mathematical thinking [**6**].

Algebra evolved, in a sense, in an effort to solve equations [**3]**. Emphasis on solving equations with only the natural numbers eventually leads to the introduction of negative numbers. Subsequently, solving equations with integers extended to the introduction of rational numbers in the form of p/q with p and q ($\neq 0$) integers. Finally, recognizing that many equations cannot be solved under the real number system, the complex numbers system is introduced. Through building the number system in this manner, students are guided through algebraic reasoning and develop number sense by engaging in counting activities where number patterns prevail.

Experts in math instruction have great concern about how the number system is currently taught to students. With addition modeled by combining sets, multiplication described as “iterated addition,” [**2**, p. 10] fractions and decimals introduced as a “by product of the division algorithm” [**2**, p. 10], students are often left with gaps in conceptual understanding when left to interpret results when multiplying by π or some other irrational number. A traditional course in algebra includes lessons in manipulation of variables and simplifying algebraic expressions, pattern identification and generalizations, mathematical structures, problem solving and solving equations [**10**]. Hence, many recommend a change in current pedagogical practices when teaching the number system and its operations, setting the stage for algebraic thinking at an early age for most students.

Bass’ recommended method for teaching the number system starts with teachers defining multiplication of real numbers in terms of the area of rectangles, and a geometric model [**2**, p. 11]. Currently, students are taught using number lines and base ten system which fails to provide students a strong conceptual understanding of the number system. By including geometric representations, teachers can clearly demonstrate not only multiplication of real numbers, but the distributive property as well.

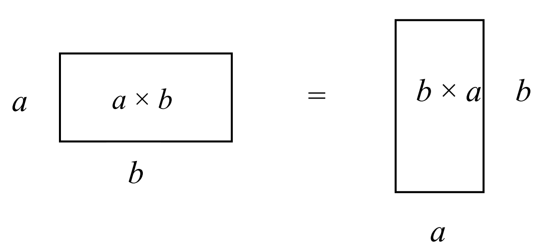

Figure 3: Geometric Model of Commutative Property [**2**, p. 12].

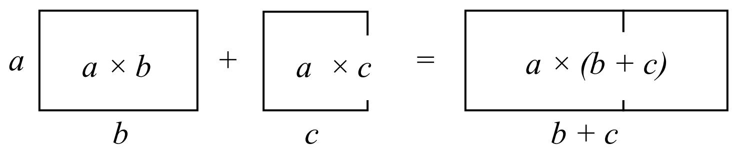

Figure 4: Geometric Model of Distributive Property [**2**, p. 12].

Not only does this approach remove much of the confusion connected to current practices, but it can easily proceed onto transformations, proportionality and guide students through demystifying such formulas as $(-a)(-b) = ab$ as a simple "double reflection" [**2**, p. 12].

By using a geometric model for real numbers, meaningful conversations empower students in early learning of "…arithmetic operations as geometric transformations" [**2**, p. 12] that will eventually lead to an easier transition into algebraic reasoning and problem solving. This philosophy supports the requirements of the NCTM of using various models to develop understandings of number system, but contradicts their recommendation of using the base 10 approach [**10**].

Modeling number systems using rectangles allows students to see the mathematics developing and have an easier time understanding processes and relationships. Using geometrical models enhances pattern recognition, proportional reasoning, and transformations which supports the functions based approach to teaching algebra while strengthening student connections. Indeed, there is no need to wait until students enroll in algebra courses to teach students fundamental elements of algebra.

Early implementation in elementary schools of teaching practices that enhance conceptual understanding and algebra readiness is critical and can ensure stronger algebra students in the future.

Chapter 6: Instructional Approaches to Increasing Algebra Readiness

There exists two particular schools of thought as to "…what mathematics should be taught and how" [**1**, p. 1]. On one side, there exists the notion that mathematics helps students develop logic, as it considers mathematical thinking as a more significant element of mathematics instruction. On the other side, are those who focus more on the algebraic ideas and focus more on content and rigor when teaching algebra? Both sides provide important elements to teaching algebra.

The importance of making connections in lieu of "factoids of information" [**1**, p.1] and problem solving skills are instrumental in helping students think algebraically. Using techniques such as "guess and check, … working backwards, use a model, solve a simpler problem, etc." [**1**, p. 1] help students solve a variety of problems effectively. In the real world, there is a need for exploring, using multiple approaches, and devising strategies to solve multiple problems.

Representing relationships visually using different forms (graphs, pictures, and diagrams), numerically (tables or lists), or verbally, provides students with a variety of ways of justifying their reasoning and extending their thought processes. According to Kriegler, "…the ability to create, interpret, and translate among representations gives students powerful tools for mathematical thinking" [**1**, p. 3]. Finally, another key component in improving algebraic thinking involves inductive reasoning, where patterns and relationships are recognized, used to draw conclusions, and applied to solve new yet similar problems [**1**].

Pedagogical practices that help strengthen algebraic thinking are based upon building conceptual understandings in elementary and middle school grades. Unfortunately, some attempts at making algebra "easy to learn", fail to help students gain a strong understanding of fundamental concepts that would more adequately prepare students for algebraic reasoning. An example of this is the "means-extremes" [**1**, p. 3] procedure for solving proportions. Though the technique provides students with an easy algorithm, it fails to "…help students understand the role of multiplication property of equality in solving equations or develop sense making notions about proportionality" [**1**, p. 3].

Instructional approaches which emphasize the arithmetic structures have been linked to increases in algebraic thinking [**3**]. In this approach, students use pictorial models to explain their reasoning, thereby making connections between their arithmetic and algebraic thinking. In fact, students who "received simultaneous instruction in both algebra and arithmetic, with an emphasis on the structural aspects of arithmetical expressions, performed better on certain algebra tasks [**3**, p. 163].

When learning the basics of algebraic symbols and syntax, Ausubel and Robinson [**1**] identified similar problems in students as when learning a second language. In order to understand the language of algebra, the concept of variable, variable expressions and the significance of the solutions must be understood [**6**, p. 4]. "The learner begins by translating algebraic symbols into the 'native' [**1**, p. 377] language of arithmetic and depends on the knowledge of arithmetic syntax in order to understand the syntax of algebra" [**1**, p. 377].

Algebra as a language refers to the ability students must have to manipulate variables, evaluate expressions, solve equations, and interpret solutions. Being able to distinguish the symbol 149 from 14*x* requires a bit of practice and instruction. Integrating real life scenarios in mathematics instruction can help students identify the relevance of algebra to their lives. Two particular problems, "Smart shopping" and "The Garden Problem" (see Figure 5) [**6**, p. 5] will be shared to more clearly explain this phenomenon.

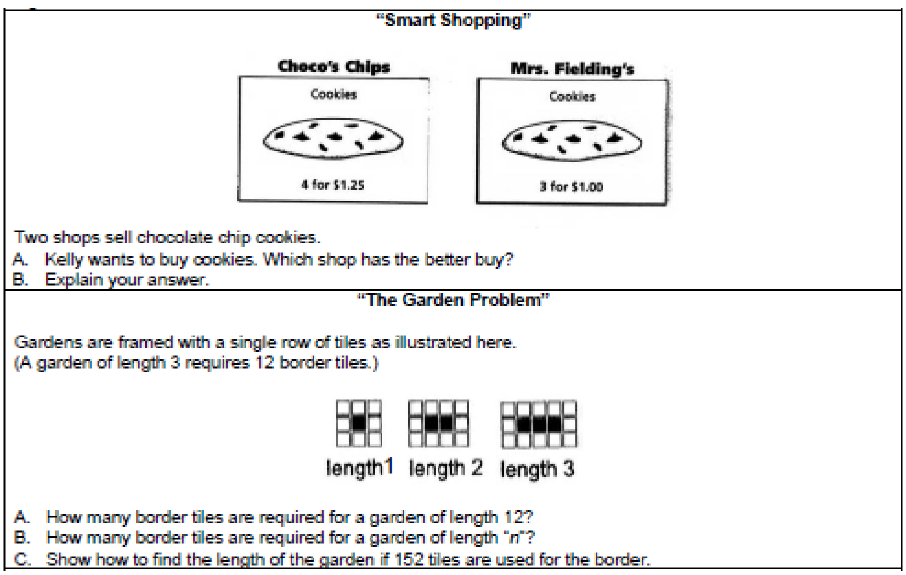

Figure 5: Sample problems [**6,** p. 5].

As we review the solution paths taken by students, much can be revealed about their conceptual understanding, thought processes and solution strategies (see Figure 6 and Figure 7).

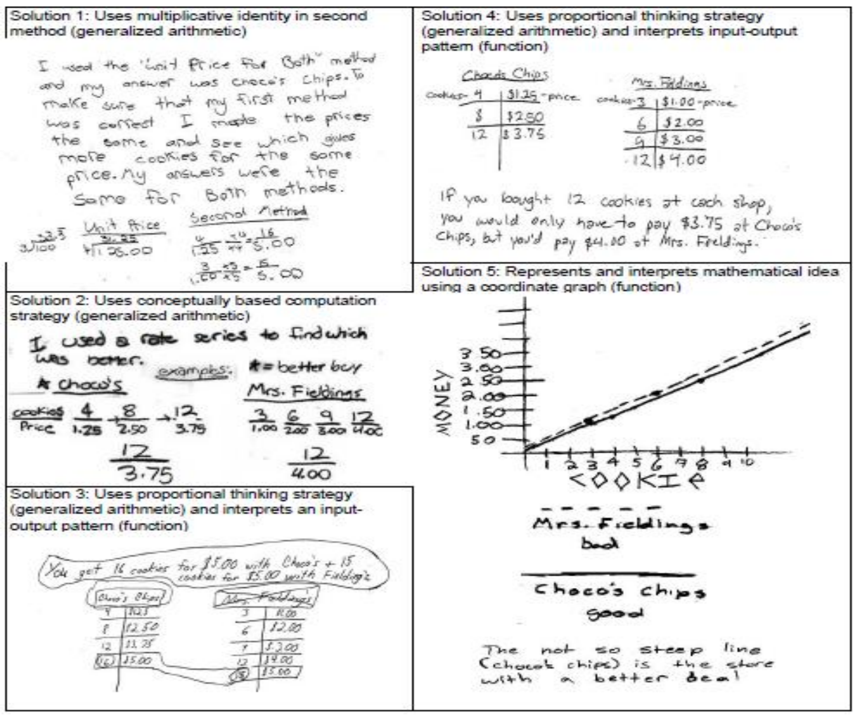

Figure 6: Solutions to "Smart Shopping" Part A [**6**, p. 6].

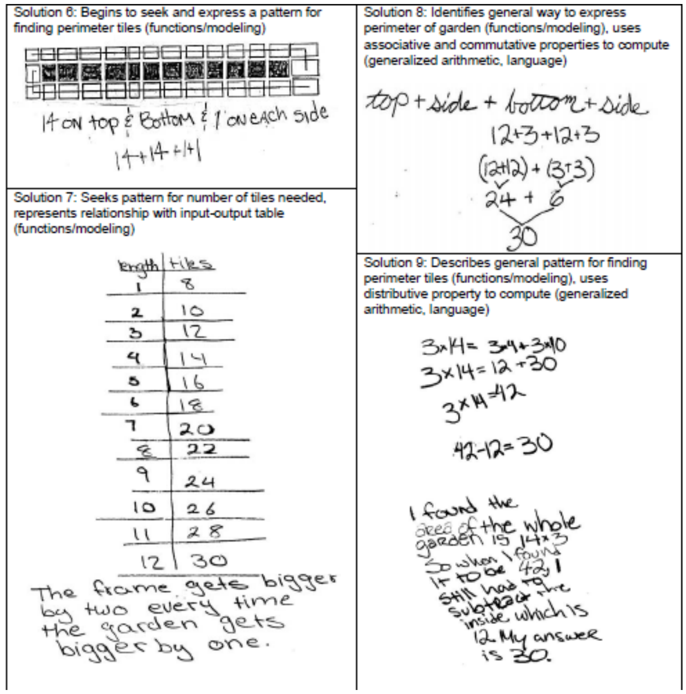

Figure 7: Solutions to "The Garden Problem" Part A [**6**, p. 7].

Student solutions to "Smart Shopping" (Figure 4), illustrated a variety of algebraic reasoning strategies including: using multiplicative identity, equivalent fractions, some proportionality, and even the use of tables to analyze input and output. In "The Garden Problem" (Figure 5), informal ideas include identifying patterns, using a variable to represent and algebraic expression, and applying "…functional relationships for the number of tiles in the geometric design" [**6**, p. 8]. All of these skills are in alignment with the NCTM standards especially as students "…represent and analyze patterns and functions, using words, tables, and graphs" [**10**, Algebra, para 3].

One approach for teaching fundamental concepts and establishing algebraic reasoning for students in elementary grades and beyond includes a unique recommendation to alter current practices in elementary schools from teaching operations on the real line to applications to using geometric representations. This brings to mind strong arguments for greater accessibility, conceptual understanding and algebra readiness in elementary age students. By changing current teaching methods during these early years, educators can help students establish deeper roots, stronger connections, and enhanced levels of algebraic reasoning as they progress to their middle and high school years [**3**].

Implications related to secondary mathematics instruction, include the impact of how some of the "easy math" techniques or so called short cuts negatively affect students' conceptual understanding of basic mathematical concepts. In efforts to make math easy for students, many teachers fail to help students understand the reasoning behind the process, hence students suffer. Instead, teachers should turn their attention to building conceptual understanding, teaching appropriate vocabulary, and enhancing skills that strengthen algebraic thinking at the early ages is essential to appropriately prepare students for success in an algebra course. The message is clear, and elementary through secondary teachers must align their standards, vocabulary, and approaches to enhance the level of instruction in the classroom and prepare students for their first algebra course the day they first enter school. No more mixed messages, no more tricks. Only through teaching the concepts can all students have strong algebraic foundations, understanding, and skills necessary to succeed throughout their entire mathematics education experience.

CONCLUSION

Algebra is recognized as the gateway course to high school graduation and post secondary education. Skills such as problem solving, critical thinking and logical reasoning are essential for success in today's world. For student's currently enrolled in schools to be competitive in the future, schools need to begin early. From elementary grades until the time students enroll in their first formal algebra course, algebraic thinking must be integrated into the curriculum. Since there are no quick or easy fixes, teachers must focus on teaching the concepts, building academic vocabulary and establishing in their students the need to think logically.

Building a strong conceptual understanding of arithmetic and making connections to algebra students can develop their algebraic reasoning using mathematical thinking tools. As conceptual understanding becomes a key component to teaching mathematics students will become better equipped to answer the call to be algebra ready. Schools and educators must combine strengths in pedagogy and math instruction to arm the students of today with the mathematical tools and knowledge needed to succeed in tomorrow's corporate world. Earlier introduction to algebra should not be a cause of fear or debate in today's schools. Instead, the challenge to prepare all students for a challenging course in algebra needs to be overcome.

Logic, problem solving and generalizations are all skills that will benefit students in future educational and career pursuits. This being the case, teachers should combine their efforts and learn best practices to lay the foundation for algebra readiness that will lead to success not only algebra, but in tomorrow's job market. There needs to be no further delay. All students should have the benefit and opportunity to learn and succeed in an algebra course. Only in this manner, will schools be able to say they have adequately provided students with the opportunity and education to succeed in the world upon graduating from high school.

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Vita

Carmen De Las Mercedes Finch was born in Havana, Cuba. She immigrated with her mother to the United States in August 1970, only a month before she turned 2. She received her Bachelor of Arts Degree from Brigham Young University in August 1990 in Mathematics Education. She was nominated and won the "Sallie Mae National First Year Teacher of the Year Award" in August 1991, one of 2 selected for the State of California. Since then, she has taught in Georgia, Utah, and most recently, in the state of Texas. Carmen served as a Secondary Mathematics Specialist for Austin Independent School District where she presented several professional development training sessions pertaining to struggling learners, ESL strategies, building academic vocabulary in the mathematics classroom, and creating successful learning environments. She has been actively involved in establishing Professional Learning Communities, and working with high need campuses as they strive to help all students succeed. She is the proud mother of five children: Austin, Skyler, Jordan, Mackenzie, and Madison. Upon completion of her Masters in Math Education at the University of Texas-Austin, Carmen will begin her doctoral studies at the University of Texas-Austin in pursuit of a Ph.D. in Higher Education Administration.

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